<u>Chapter 1</u>

Introduction to Operations Research

Operations Research: A Practical Introduction

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Learning Objectives

- Learn about the origins and applications of operations research
- Understand system modeling principles
- Understand algorithm efficiency and problem complexity
- Contrast between the optimality and practicality
- Learn about software for operations research

1.1 The origins and applications of operations research

- **Operations Research** can be defined as the use of quantitative methods to assist analysts and decision-makers in designing, analyzing, and improving the performance or operation of **systems**. A system can be:
  - *A manufacturing system*
  - *Engineering System*
  - *Financial*
  - *Service*
  - *Many others*
- The field of Operations Research incorporates analytical tools from many different disciplines, so that they can be applied in a rational way to help decision-makers solve problems and control the operations of systems and organizations in the most practical or advantageous way.
- The ideas and methodologies of Operations Research have been taking shape throughout the history of science and mathematics, but most notably since the industrial revolution.
- Operations Research (also called **Management Science**) became an identifiable discipline during the days leading up to World War II. In the 1930s, the British military buildup centered around the development of weapons, devices, and other support equipment.

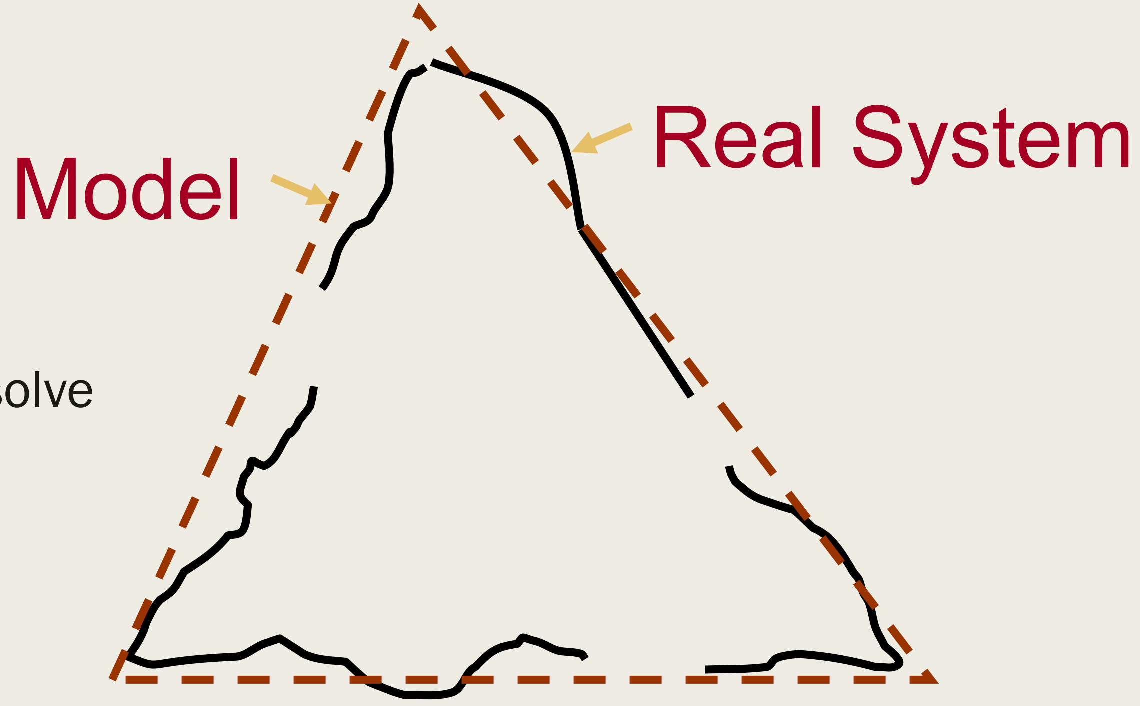
1.2 System Modeling Principles

- A **model** is a simplified, idealized representation of a real object, a real process, or a real system.
- The models used are called **mathematical models** because the building blocks of the models are mathematical structures (equations, inequalities, matrices, functions, and operators).
- Step in building a model:

    1. Discover an area that is in need of study or improvement.
    2. Determine which aspects of the system are controllable and which are not
    3. Identify the goals or purpose of the system,
    4. Identify the constraints or limitations that govern the operation of the system
    5. Create a model that implicitly or explicitly embodies alternative courses of action
    6. Collect data that characterize the system being modeled.
    7. Solve the model and conduct your analysis

The Art and Science of Modeling

- The best model is one that strikes a practical compromise in representing a system as realistically as possible, while still being understandable and computationally tractable.
- Detail ≠ Accuracy
  - *Not all details are correct*
  - *Not all details are relevant*
- If the model is unnecessarily too detailed, it may become too complex to analyze or solve
- It is possible to build models that are both realistic and simple



Practical Advice before modeling and solving problems

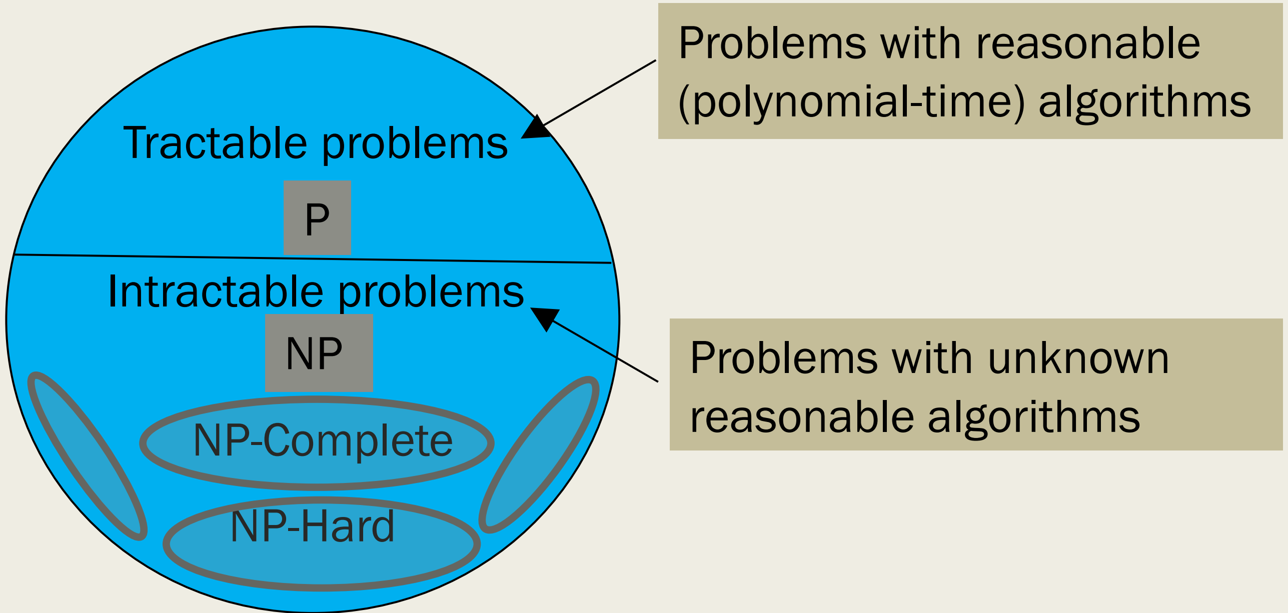
- Does the problem need to be solved?
- Will it be possible to determine what the *real* problem is?
- If a model were developed and a solution proposed, would anybody care?
- Would anybody try to implement the solution?
- How much of the analyst's time and expense is it worth to try to solve this problem?
- Is there enough time and adequate resources available to make any significant progress toward solving this problem?
- Will the solution create other serious problems for which there is no apparent remedy?

1.3 Algorithm Efficiency and Problem Complexity

- An **algorithm** is a sequence of operations that can be carried out in a finite amount of time.
- An algorithm may be called repeatedly or recursively but it will eventually terminate.
- Several factors influence the amount of time it takes for a computer program to execute to solve a problem:
  - *the programming language used,*
  - *the programmer's skill,*
  - *the hardware used in executing the program,*
  - *and the task load on the computer system during execution.*
- None of these factors is a direct consequence of the underlying **algorithm** that has been implemented
- The performance of an algorithm is often described as a function of **problem size**, which denotes the size of the data set that is input to the algorithm.
  - ***Example:*** *Sorting algorithm takes longer to sort a list of 10,000 names than to sort a list of 100 names.*

Two Classes of Problems (from Complexity Perspective)

- **Class P:** problems that can be solved by an algorithm within an amount of computation time proportional to some polynomial function of problem size; that is, the problems are solvable by **polynomial-time algorithms**
- **Class NP** (**N**ondeterministic **P**olynomial time) contains problems that may require computation time proportional to some exponential (or larger) function of problem size; these algorithms are called **exponential-time algorithms.**



Algorithm Efficiency

- How do we evaluate of the **performance of algorithms** without being dependent on software/hardware/developer skill/etc?
  👉 *Number of computational steps*
- Should we use:
  - *Best Case? Too optimistic*
  - *Average Case? Needs statistical assumptions and analysis*
  - ***Worst Case**? Most practical*
- **Worst case** : The greatest number of steps that may be necessary for guaranteed completion of the execution of the algorithm.

Example: ***M**ultiplying two $n \times n$ matrices takes, in the worst case, time proportional to n^3*

Big-Oh notation (Order-of)

- An algorithm is said to be O(f(n)) if there exist constants c and n_0 such that for all $n > n_0$, the execution time of the algorithm is $\leq c \cdot f(n)$.
- n denotes problem size and f(n) is some function of problem size.
- The function f(n) is the algorithm's worst case step count, measured as a function of the problem size.
- The constant c is called a constant of proportionality, and is intended to account for the various extraneous factors that influence execution time, such as hardware speed, programming style, and computer system load during execution of the algorithm.
- A computational step can be a comparison, multiplication, addition, etc.
- <u>Example</u>: If an algorithm takes a maximum of n^2 steps, its efficiency will then be $O(n^2)$, where the “O” stands for “Order of”

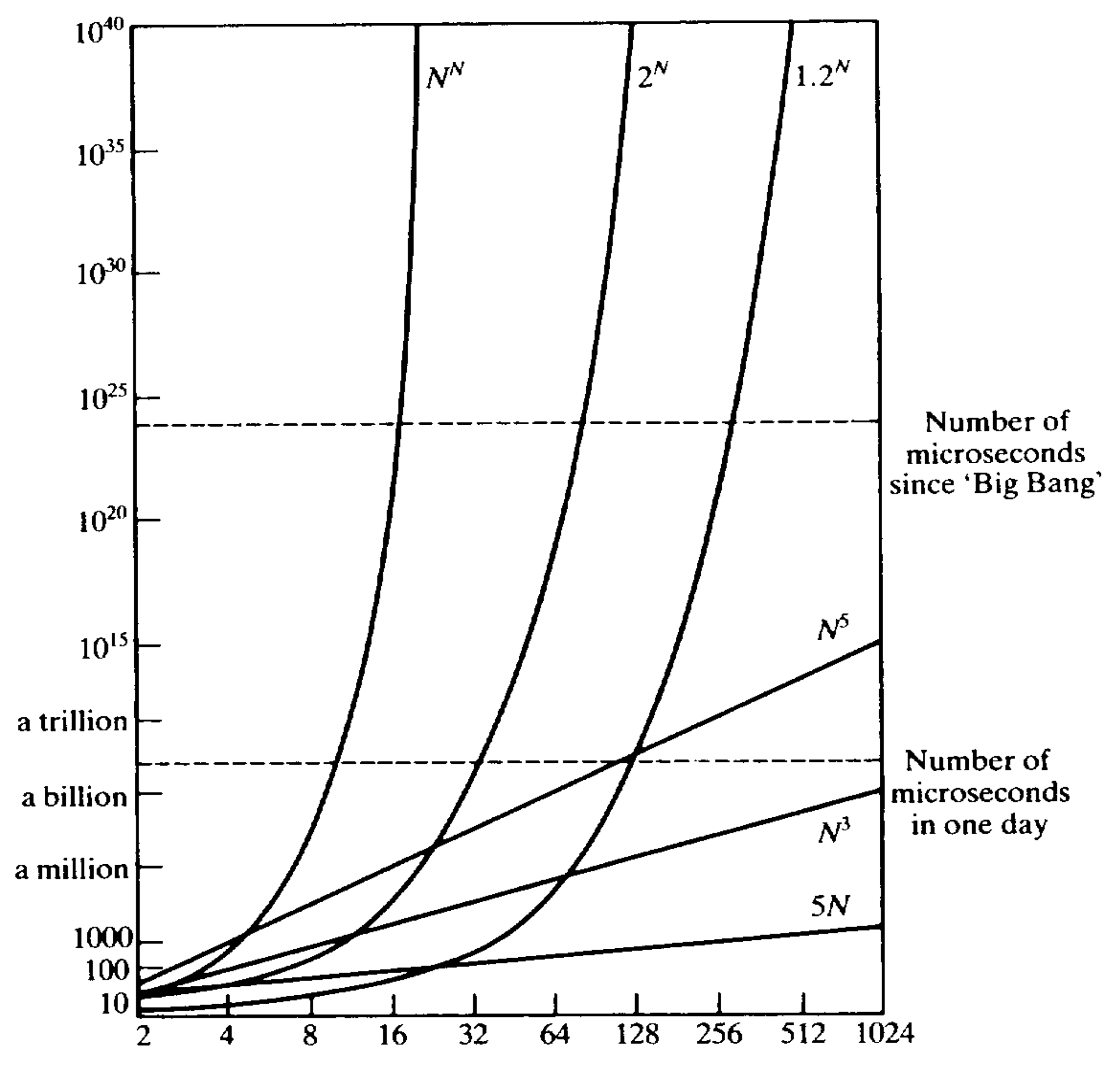
≈

Computational times as a function of problem size

TABLE 1.1
Computation Times

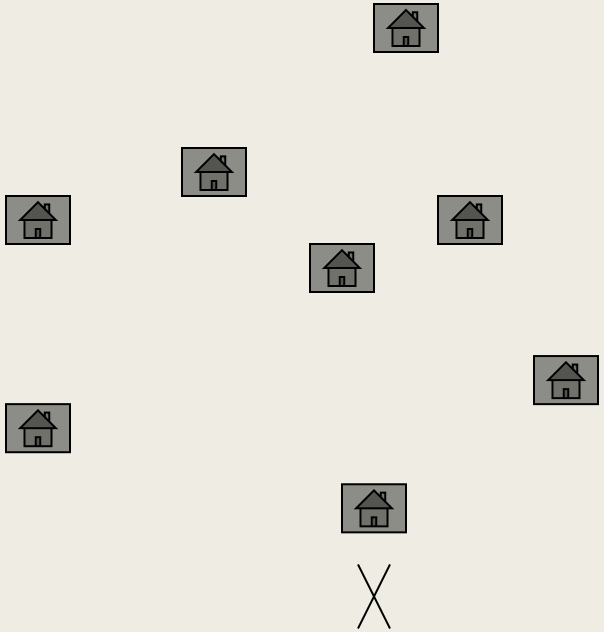
f(n)	n = 10	n = 20	n = 50	n = 100
n	10 s	20 s	50 s	100 s
n^2	100 s	400 s ≈ 7 min	2500 s ≈ 42 min	10,000 s ≈ 2.8 h
n^3	1000 s ≈ 17 min	8000 s ≈ 2 h	125,000 s ≈ 35 hrs	1,000,000 s ≈ 12 d
2^n	1024 s ≈ 17 min	1,048,576 s ≈ 12 d	1.126×10^{15} s ≈ 350,000 centuries	1.268×10^{30} s ≈ 10^{21} centuries
$n!$	3,628,8 s ≈ 1 month	2.433×10^{18} s ≈ 10^9 centuries	3.041×10^{64} s ≈ 10^{55} centuries	
n^n	10^{10} s ≈ 300 years	1.049×10^{26} s ≈ 10^{17} centuries	8.882×10^{84} s ≈ 10^{75} centuries	

Computational times function as of problem size



Harel, D. (1992) Algorithmics: The Spirit of Computing (2nd Edition)

Search Space Complexity:
The Traveling Salesman Problem (TSP)

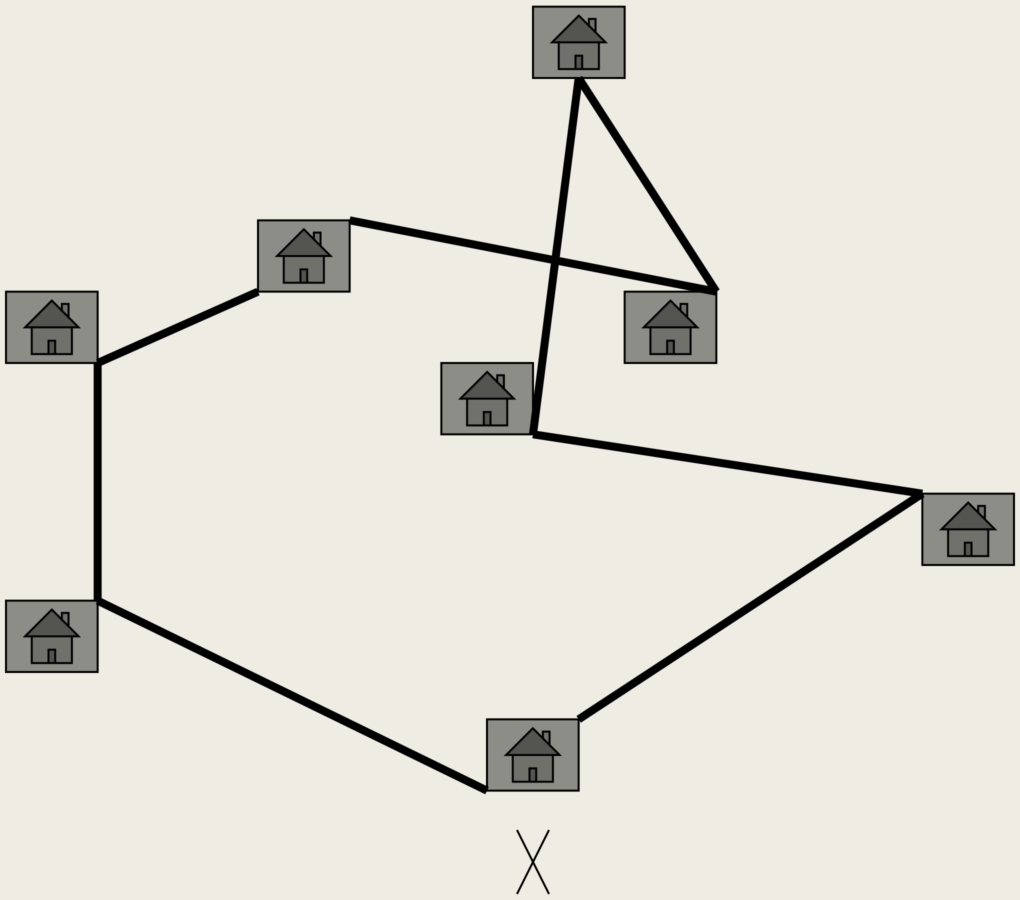


- A salesman must visit *n* cities
- Each city must be visited only once

Problem: Which path should the salesman take to minimize the total *distance* traveled ? (i.e., which tour has shortest distance?)

Distance here can represent cost (can be time, fuel, etc)

The Traveling Salesman Problem (TSP)



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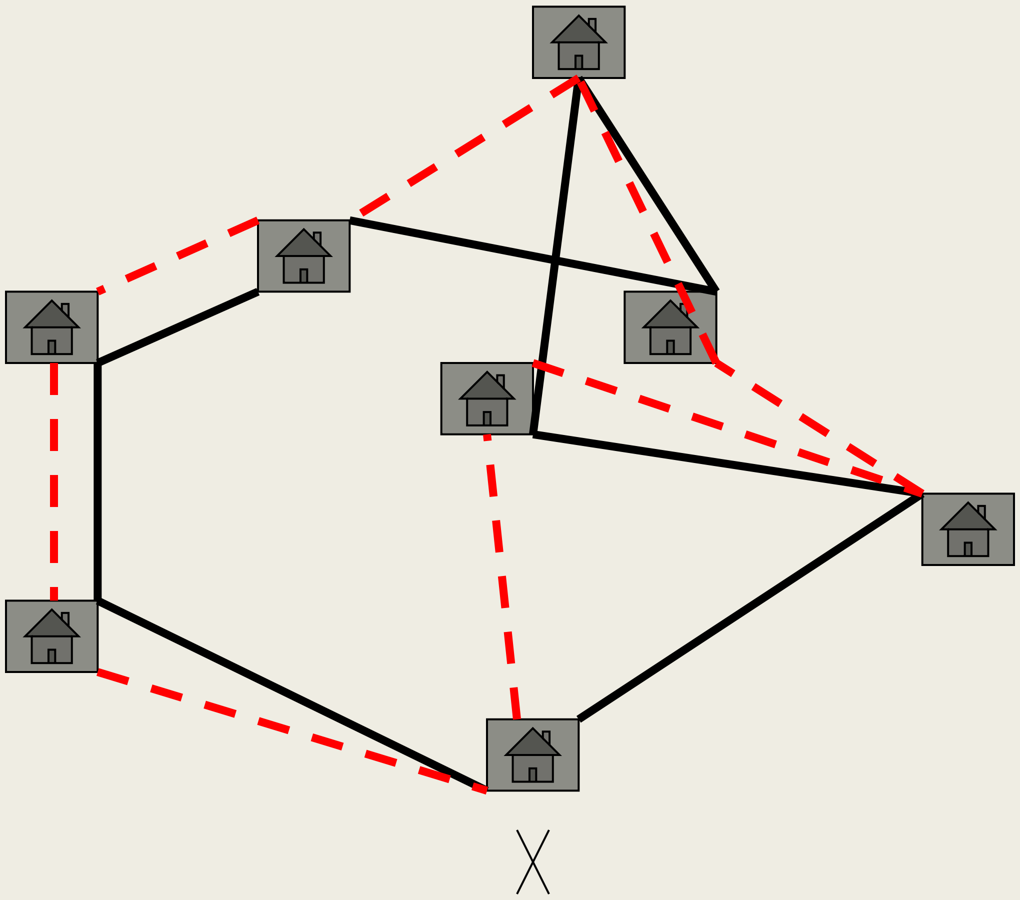

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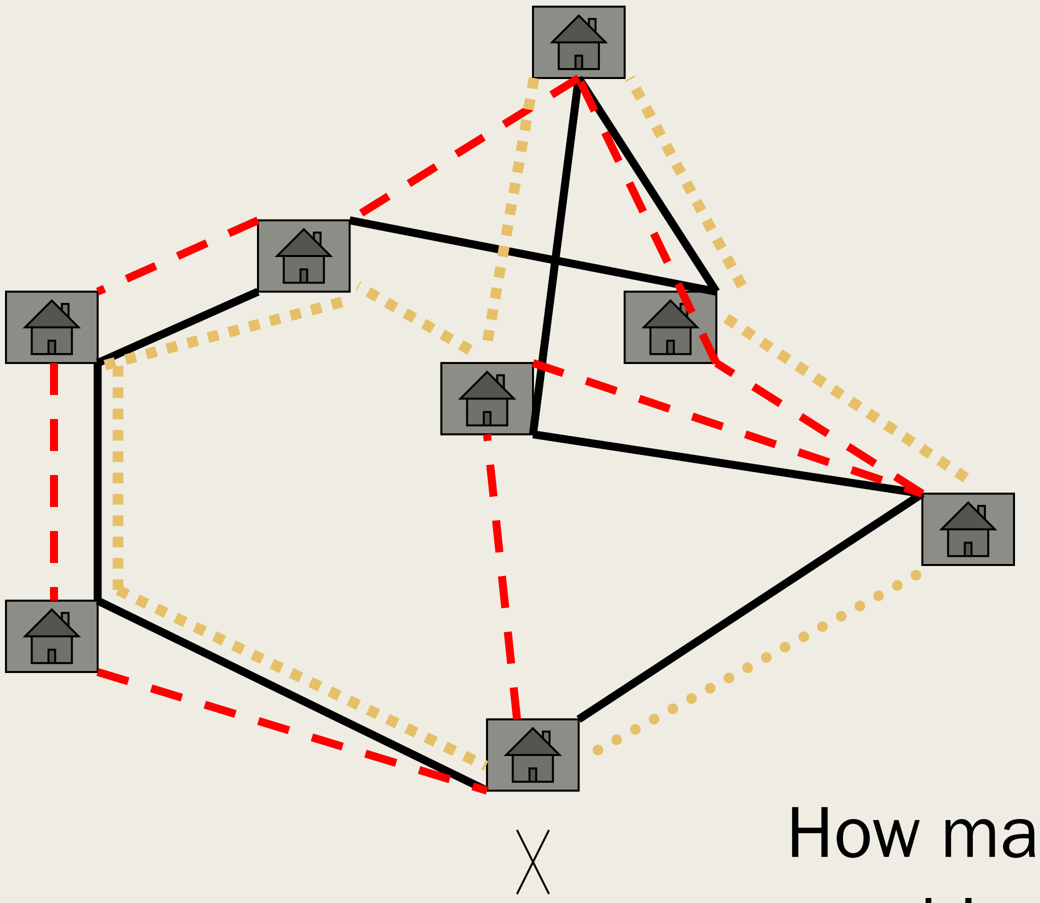

How many possible combinations?

The TSP

- Try a small example by hand
    - *A TSP instance of n = 4*

dij	1	2	3	4
1		10	20	5
2	15		35	10
3	15	10		40
4	5	20	20	

Solving The TSP

- If we enumerate all possibilities, the number of combinations is $(n - 1)!$. That is, the size of the search space $= |S| = (n - 1)!$
- Suppose we have a computer that evaluates a 1000 solution every second
- How long does it take to find the **optimal** solution for a TSP instance with 10, 12, 17, 20, 24 cities if all possibilities are enumerated?

Full Enumeration of The TSP

n	**Computer Time** (at 1000 solutions per sec)
6	0.12 sec
8	5.04 sec
10	6.05 min
12	11.09 hour
14	72.07 days
16	41.47 years
18	11278.77 years
20	3,857,341 years
22	1,620,083,149 years (1.62 billion years)
24	819,762,073,200 years (819.76 billion years)

Even if computers become much faster

n	**Computer Time**		
	10^3 sol/sec	**10^6 sol/sec**	**10^9 sol/sec**
10	6.05 min	0.63 sec	
14	72.07 days	1.73 hours	
18	11278.77 years	11.28 years	
24	819.76 billion years	819.76 million years	
26		491.86 billion years	491.86 million years

1.4 Optimality and Practicality

- We have been trained to search for exact (perfect) solutions to problems using mathematics, but this is not always possible for various reasons:

👉 Models are approximate representations of the real system. So, even if we obtain exact solutions to the model, such solutions would not necessarily constitute exact solutions to real system.

👉 Automatic computing devices cannot store exact representation of real numbers, so numbers are often rounded off. **Accumulated round-off error** can distort the final results.

👉 In many cases, input data are approximated values, which means exact solutions are not perfect for the exact problem

👉 Finally, the inherent difficulty of some problems might suggest that accepting suboptimal solutions is the only practical approach especially for algorithms that take an exponential amount of computation time to guarantee optimal solutions.

- Settling for solutions of merely “good enough” is not always settling for lower standards especially in a complex and subjective world.

1.5 Software for operations research

Common modeling environments	Common solvers	Software libraries
Excel Spreadsheets Google Sheets AMPL MPL Lingo OPL AIMMS SAS/OR OPTMODEL GAMS NEOS	Frontline Solver Cplex Gurobi	Google OR-Tools COIN-OR (open source) IMSL Math Subroutine Library

A comprehensive list is available at:

https://en.wikipedia.org/wiki/List_of_optimization_software